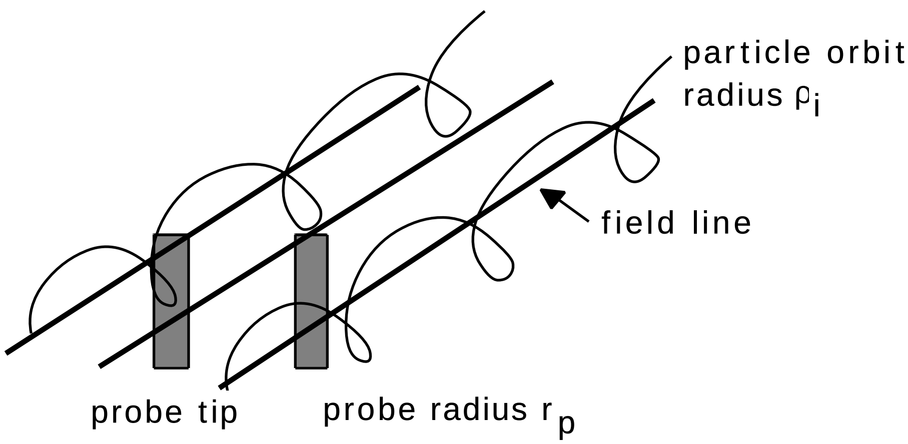
Gaussian Statistics

Contents

VOLUME SAMPLING DIAGNOSTICS

The Langmuir probe

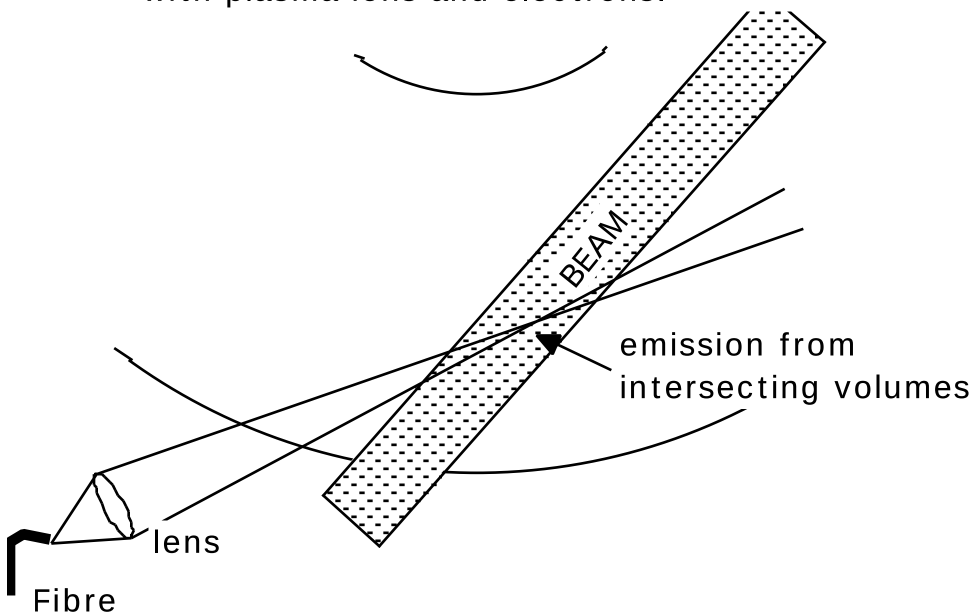
Restrictions: sample vol. size effects, no. of sample vols.
Collects fluctuations over $2(r_p + \rho_i)$, only two sample points



Beam Emission Spectroscopy

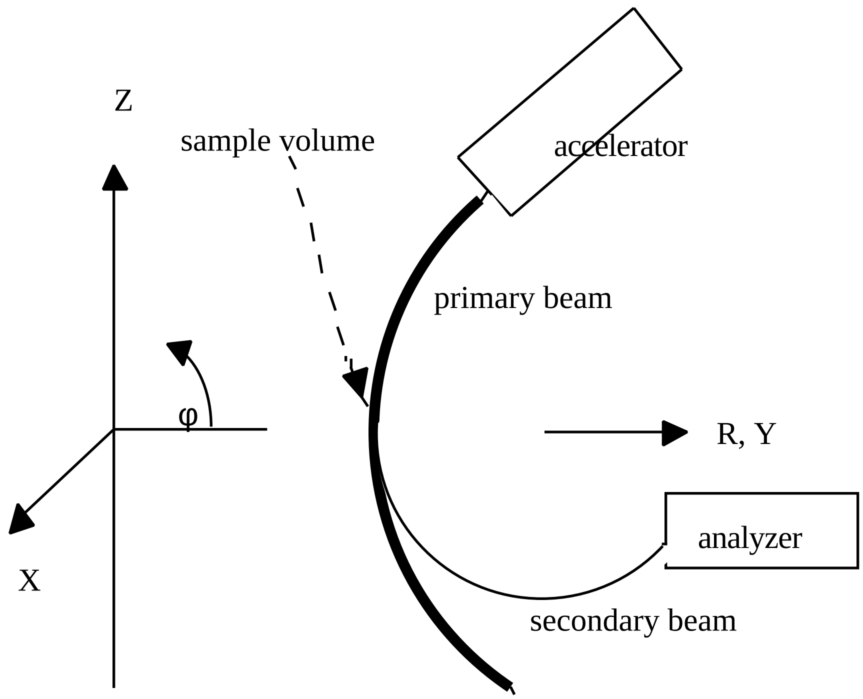
Restrictions: sample vol. size, path effects.
Collects over ≈ 2 x 2 cm., beam through fluctuating medium.

Measure light intensity fluctuations emitted from a neutral beam excited by collisions with plasma ions and electrons.

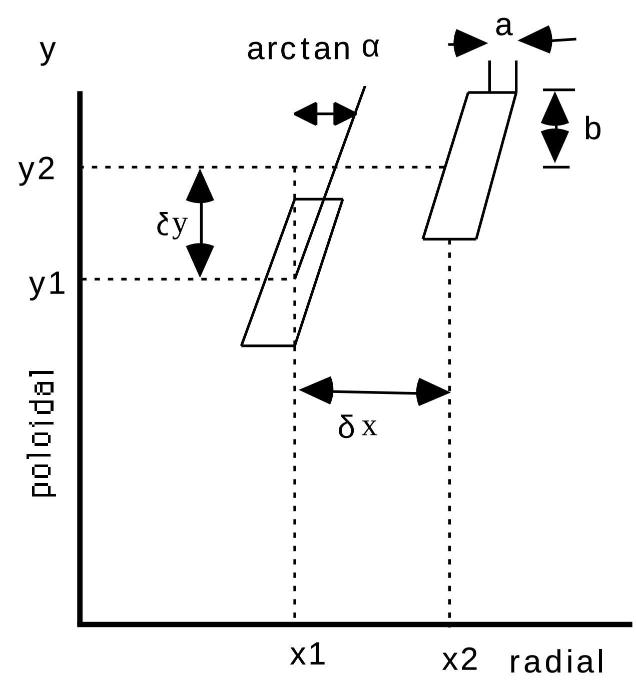


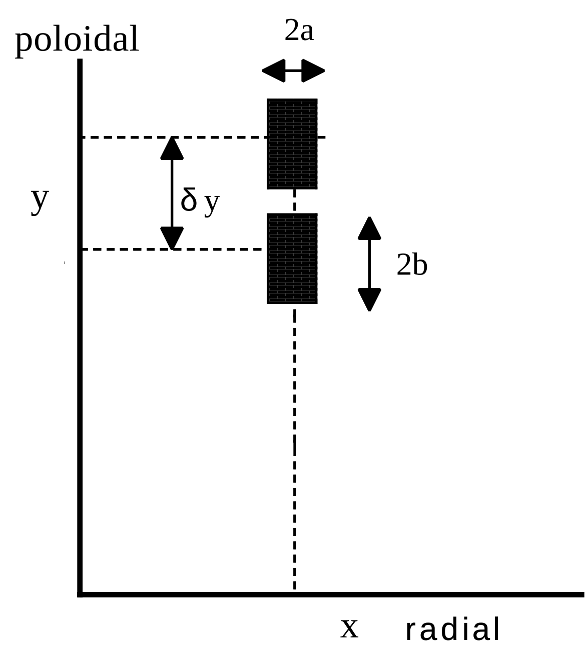
The Heavy Ion Beam probe

Restrictions: sample vol. size, no of sample vols., path effects.



FINITE SAMPLE VOLUME EFFECTS





(dx,dy) correspond to (dr,rdθ) in the cylindrical geometry. Centers separated by $(\delta x, \delta y) = (\delta r, r\delta\theta)$. Either fixed hard edges of length 2a and 2b, or Gaussian profiles with e-folding widths a and b, and orientation angle $\alpha = (rd\theta/dr)^{-1}$. Mostly consider $\alpha = 0$, $\delta x = 0$, i.e. we are measuring poloidal components of wave numbers.

For the “hard edge” case, the sample vol. average is given by

$$\bar{\tilde{n}}_e\left(x_j,y_j\right)=\frac{1}{A}\int_{y_j-b}^{y_j+b}dy\int_{x_j-a+\alpha(y-y_j)}^{x_j+a+\alpha(y-y_j)}dx\,\tilde{n}_e(x,y)$$

with the area A = 4ab. For the Gaussian profiled sample volumes, the sample volume average is given by

$$\bar{\tilde{n}}_e\left(x_j,y_j\right)=\frac{1}{A}\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dx\,e^{\left[-\frac{\left(x-x_j-\alpha\left(y-y_j\right)\right)^2}{a^2}-\frac{\left(y-y_j\right)^2}{b^2}\right]}\tilde{n}_e(x,y)$$

where the area A is

$$A=\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dx\,e^{\left[-\frac{\left(x-x_j-\alpha\left(y-y_j\right)\right)^2}{a^2}-\frac{\left(y-y_j\right)^2}{b^2}\right]}=\pi ab$$

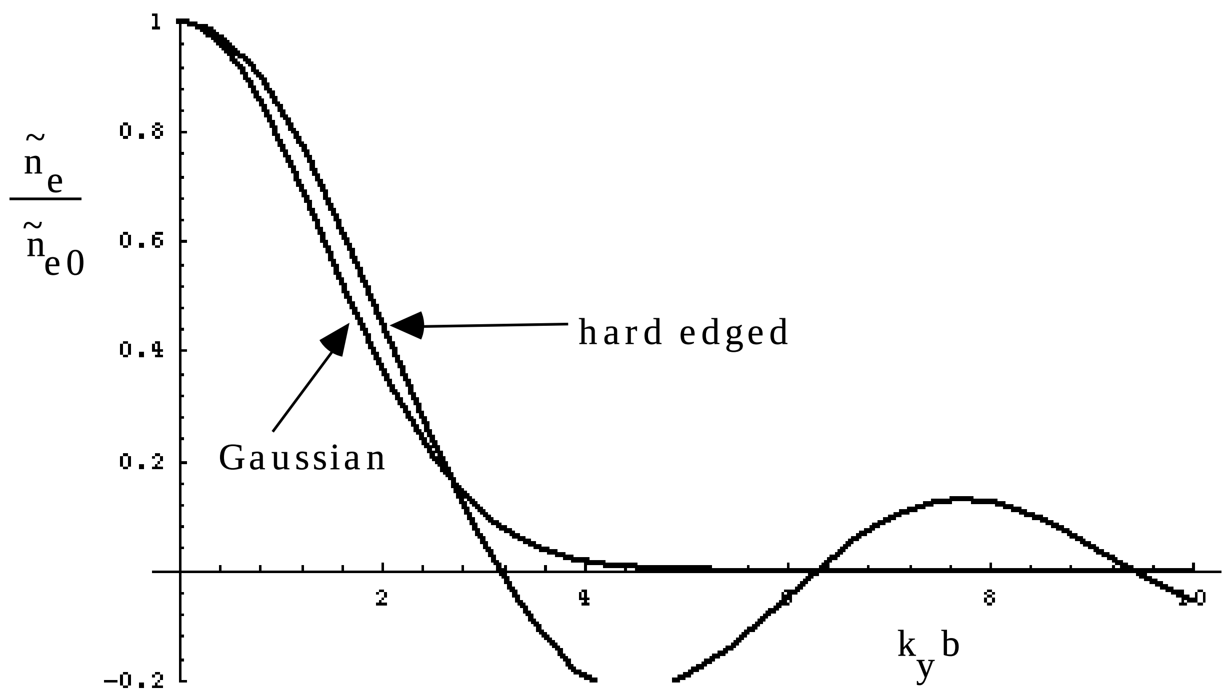
Take a simple 1D case of a wave $\tilde{n}_e=\tilde{n}_{e0}\mathrm{Cos}\left(k_y y\right)$. Then the volume averaged measurement is, for the hard edged case

$$\bar{\tilde{n}}_e=\frac{1}{2b}\int_{-b}^{b}dy\,\tilde{n}_{e0}\mathrm{Cos}\left(k_y y\right)=\tilde{n}_{e0}\frac{\mathrm{Sin}\left(k_y y\right)}{k_y y}$$

and in the Gaussian case

$$\bar{\tilde{n}}_e=\frac{1}{\sqrt{\pi}b}\int_{-\infty}^{\infty}dy\,e^{-\left(\frac{y}{b}\right)^2}\tilde{n}_{e0}\mathrm{Cos}\left(k_y y\right)=\tilde{n}_{e0}e^{-\left(\frac{bk_y}{2}\right)^2}$$

The normalized results ($\bar{\tilde{n}}_e / \tilde{n}_{e0}$) are shown below as a function of $k_y b$: we see the filtering action of each sample volume is similar for small $k_y b$, but for large $k_y b$ the "hard edge" case introduces zeros where $k_y b = n\pi$, i.e. when an exact number of wavelengths fits into the sample volume (remember the length is 2b). Clearly we should use $k_y b < 1$ to avoid errors.



Next we want to consider what happens with turbulence present. We will consider only the Gaussian sample volumes, so that the average of a Fourier component is given by

$$\bar{\tilde{n}}_{k\omega} = \int_{-\infty}^{\infty} dx\,\hat{\tilde{n}}_{\omega}(x)\,e^{i\left(k_x x + k_y y\right)}$$

$$= \frac{\hat{\tilde{n}}_{\omega}}{\pi ab}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\,e^{\left[-\frac{\left(x-x_j-\alpha\left(y-y_j\right)\right)^2}{a^2}-\frac{\left(y-y_j\right)^2}{b^2}\right]}e^{i\left(k_x x+k_y y\right)}$$

$$= \hat{\tilde{n}}_{\omega}\,e^{\left[-\left(\frac{k_x a}{2}\right)^2-\left(\frac{\left(k_y+\alpha k_x\right)^2 b}{2}\right)^2+ik_x x_j+ik_y y_j\right]}$$

The power spectrum S(k,ω) is defined by

$$S(k,\omega) = \lim_{T\to\infty}\frac{1}{T}\left|\tilde{n}_e(k,\omega)\right|^2$$

so that the effective sample volume averaged spectrum is

$$\bar{S}(k,\omega) = \lim_{T\to\infty}\frac{1}{T}\left|\bar{\tilde{n}}(k,\omega)\right|^2$$

$$= S(k,\omega)\,e^{\left[-2\left(\frac{k_x a}{2}\right)^2-2\left(\frac{\left(k_y+\alpha k_x\right)^2 b}{2}\right)^2\right]}$$

Assuming each sample volume has the same size and shape we obtain the cross power spectrum of the sample volume averages

$$
\begin{aligned}
\bar{P}_\omega(r_1,r_2) &= \lim_{T\to\infty}\frac{1}{T}\left\langle \bar{\tilde{n}}^*_\omega(r_1)\bar{\tilde{n}}_\omega(r_2)\right\rangle \\
&= \int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\int_{-\infty}^{\infty}\frac{dk_y}{2\pi}\bar{S}(k,\omega)e^{ik\cdot\delta r} \\
&= \bar{S}_\omega\bar{\gamma}_\omega(r_1,r_2)e^{i\bar{\varphi}_\omega(r_1,r_2)}
\end{aligned}
$$

The last form of the expression is a standard form, in which

$$
\bar{S}_\omega = \left[\left|\bar{P}_\omega(r_1,r_1)\right|\left|\bar{P}_\omega(r_2,r_2)\right|\right]^{\frac{1}{2}}
$$

is the mean of the cross power spectrum, $\tilde{\gamma}$ and $\bar{\varphi}$ are the coherence and phase.

Now we have to assume a form for S(k,ω): take a Gaussian

$$
S(k,\omega) = \pi l_r l_\theta S(\omega)e^{\left[-\frac{l_r^2}{4}(k_r-\bar{k}_r)^2-\frac{l_\theta^2}{4}(k_\theta-\bar{k}_\theta)^2\right]}
$$

Here the correlation lengths l_r and l_θ are defined to be one e-folding distance of the intrinsic coherence γ:

$$
\gamma_\omega = e^{-\left[\left(\frac{\delta r}{l_r}\right)^2+\left(\frac{r\delta\theta}{l_\theta}\right)^2\right]}
$$

The half widths (standard deviations) of the wave number spectrum are given by

$$
\sigma_{kr} = \sqrt{2}/l_r \text{ and } \sigma_{k\theta} = \sqrt{2}/l_\theta
$$

With these forms we obtain an expression for $\bar{P}_\omega(r_1,r_2)$:

$$\bar{P}_\omega(r_1,r_2) = l_x l_y \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{\infty} \frac{dk_y}{2\pi}$$

$$e^{\left[ik_x\delta x + ik_y\delta y - 2\left(\frac{k_x a}{2}\right)^2 - 2\left(\frac{(k_y+\alpha k_x)b}{2}\right)^2 - \frac{l_x^2}{4}(k_x-\bar{k}_x)^2 - \frac{l_y^2}{4}(k_y-\bar{k}_y)^2\right]}$$

This can be written in the usual form if

$$\frac{\bar{S}(\omega)}{S(\omega)} = \frac{1}{\sqrt{h}}\,\mathbf{exp}\left[-\frac{\left(\bar{k}_y+\alpha\bar{k}_x\right)^2 b^2 + a^2\left[\bar{k}_x^2\left(1+\frac{2b^2}{l_y^2}\right)+\bar{k}_y^2\frac{2b^2}{l_x^2}\right]}{2h}\right]$$

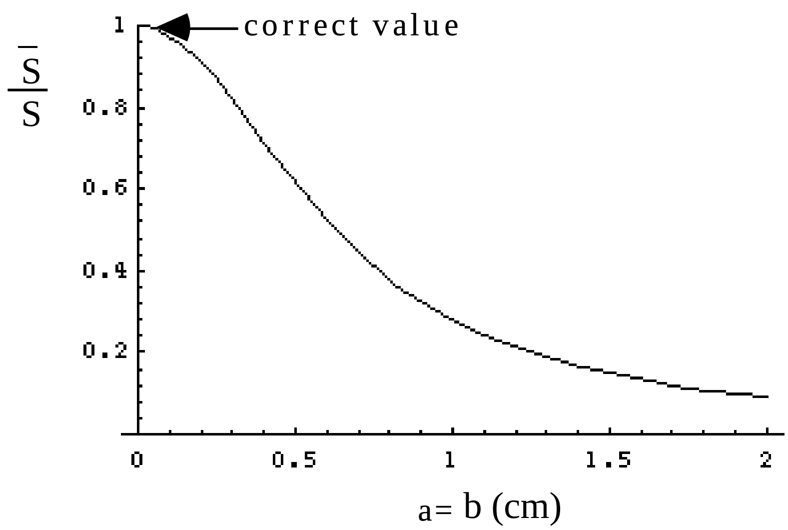
$$\bar{\gamma}_\omega = \mathbf{exp}\left[-\frac{\left(\frac{\delta x}{l_x}\right)^2 + \left(1+\frac{2a^2}{l_x^2}\right)\left(\frac{\delta y}{l_y}\right)^2 + \frac{2b^2\left(\delta x-\alpha\delta y\right)^2}{l_x^2 l_y^2}}{h}\right]$$

$$\bar{\varphi}_\omega = \frac{\bar{k}_x\delta x + \bar{k}_y\delta y + \left(\frac{2b^2}{l_y^2}\bar{k}_x - \frac{2\alpha b^2}{l_x^2}\bar{k}_y\right)\left(\delta x-\alpha\delta y\right) + \frac{2a^2}{l_x^2}\bar{k}_y\delta y}{h}$$

where

$$h = 1 + \frac{2b^2}{l_y^2} + \frac{2\left(\alpha^2 b^2 + a^2\right)}{l_x^2} + \frac{4a^2b^2}{l_x^2 l_y^2}$$

We consider some examples, taking $\alpha = 0$, $\delta x = 0$, (sample volumes aligned in the poloidal direction), with a = b (equal lengths). For the turbulence we take $l_x = l_y = 1$ cm, $k_x = 0$. Figure shows the fraction of the power $(\bar{S}(\omega)/S(\omega)$ which we measure as the size of the sample volume is increased: We must use b << a correlation length to obtain correct results.

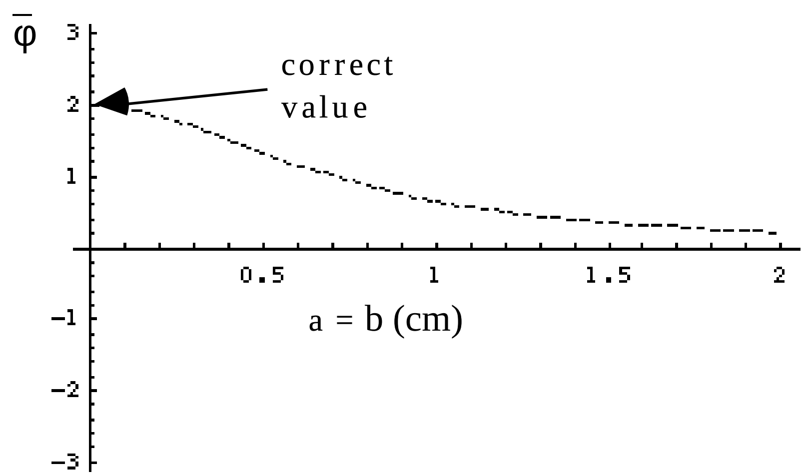


Next consider two sample volumes separated by 2 cm, and plot the observed phase as a function of sample volume size. We would deduce a wave vector

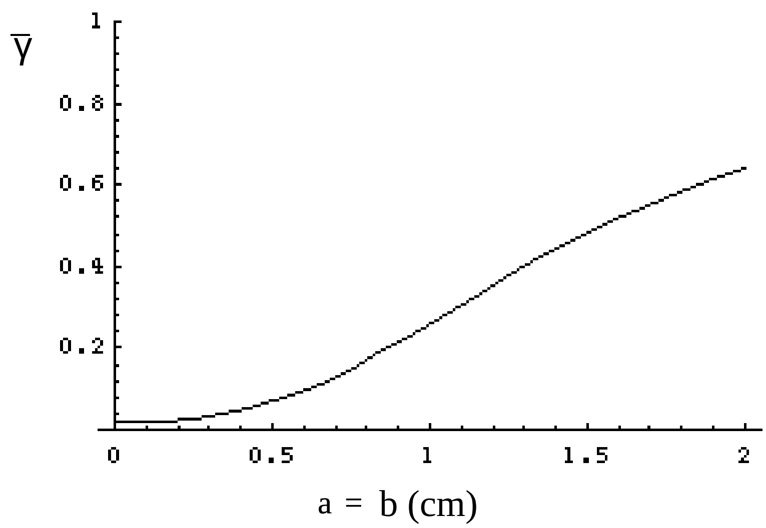
$$\bar{k}_y = \frac{\bar{\varphi}}{\delta y}$$

so that for b = 0 (no sample volume size effects) we expect $\bar{\varphi} = k_y \delta y$ = 2 cm. When a = b = l_c, the error is about 60%. Therefore we should arrange a, b << l_c to obtain accurate wave vectors

Note the ratio of the separation of the sample volumes to the sample volume size $\delta y / b$ is not important.



Lastly we look at the coherence $\bar{\gamma}$ as a function of sample volume size, b. In the example considered the sample volumes are separated by more than a correlation length (l_y = 1 cm, δy = 2 cm) so that the initial correlation is very low. As the sample volumes increase and overlap, so the correlation increases. With a = b = l_c, the error in γ is about x 10! (but an error in the computed correlation length of about 50%)



TWO FEATURES

All this is for one feature. To extend to two, we imagine the case where the individual features are uncorrelated, so that

$$\bar{P}_{\omega\Sigma}(r_1,r_2) = \lim_{T\to\infty}\frac{1}{T}\left\langle\overline{\left(\tilde{n}_{e\omega 1}(r_1)+\tilde{n}_{e\omega 2}(r_1)\right)}\left(\tilde{n}_{e\omega 1}(r_2)+\tilde{n}_{e\omega 2}(r_2)\right)\right\rangle$$

$$= \lim_{T\to\infty}\frac{1}{T}\left\langle\left(\bar{\tilde{n}}_{e\omega 1}(r_1)\bar{\tilde{n}}_{e\omega 1}(r_2)\right)+\left(\bar{\tilde{n}}_{e\omega 2}(r_1)+\bar{\tilde{n}}_{e\omega 2}(r_2)\right)\right\rangle$$

$$= \bar{P}_{\omega 1}(r_1,r_2)+\bar{P}_{\omega 2}(r_1,r_2)$$

$$= \bar{S}_{\omega 1}\bar{\gamma}_{\omega 1}(r_1,r_2)e^{i\bar{\varphi}_{\omega 1}(r_1,r_2)}+\bar{S}_{\omega 2}\bar{\gamma}_{\omega 2}(r_1,r_2)e^{i\bar{\varphi}_{\omega 2}(r_1,r_2)}$$

We want to write this in the form

$$\bar{P}_{\omega\Sigma}(r_1,r_2) = \bar{S}_{\omega\Sigma}\bar{\gamma}_{\omega\Sigma}(r_1,r_2)e^{i\bar{\varphi}_{\omega\Sigma}(r_1,r_2)}$$

which is arranged if

$$S_{\omega\Sigma} = S_{\omega 1}+S_{\omega 2}$$

$$\varphi_{\omega\Sigma} = \frac{S_{\omega 1}\gamma_{\omega 1}Sin(\varphi_{\omega 1})+S_{\omega 2}\gamma_{\omega 2}Sin(\varphi_{\omega 2})}{S_{\omega 1}\gamma_{\omega 1}Cos(\varphi_{\omega 1})+S_{\omega 2}\gamma_{\omega 2}Cos(\varphi_{\omega 2})}$$

$$\gamma_{\omega\Sigma} = \left(\frac{(S_{\omega 1}\gamma_{\omega 1})^2+(S_{\omega 2}\gamma_{\omega 2})^2+2S_{\omega 1}S_{\omega 2}\gamma_{\omega 1}\gamma_{\omega 2}Cos(\varphi_{\omega 1}-\varphi_{\omega 2})}{(S_{\omega 1}+S_{\omega 2})^2}\right)^{\frac{1}{2}}$$

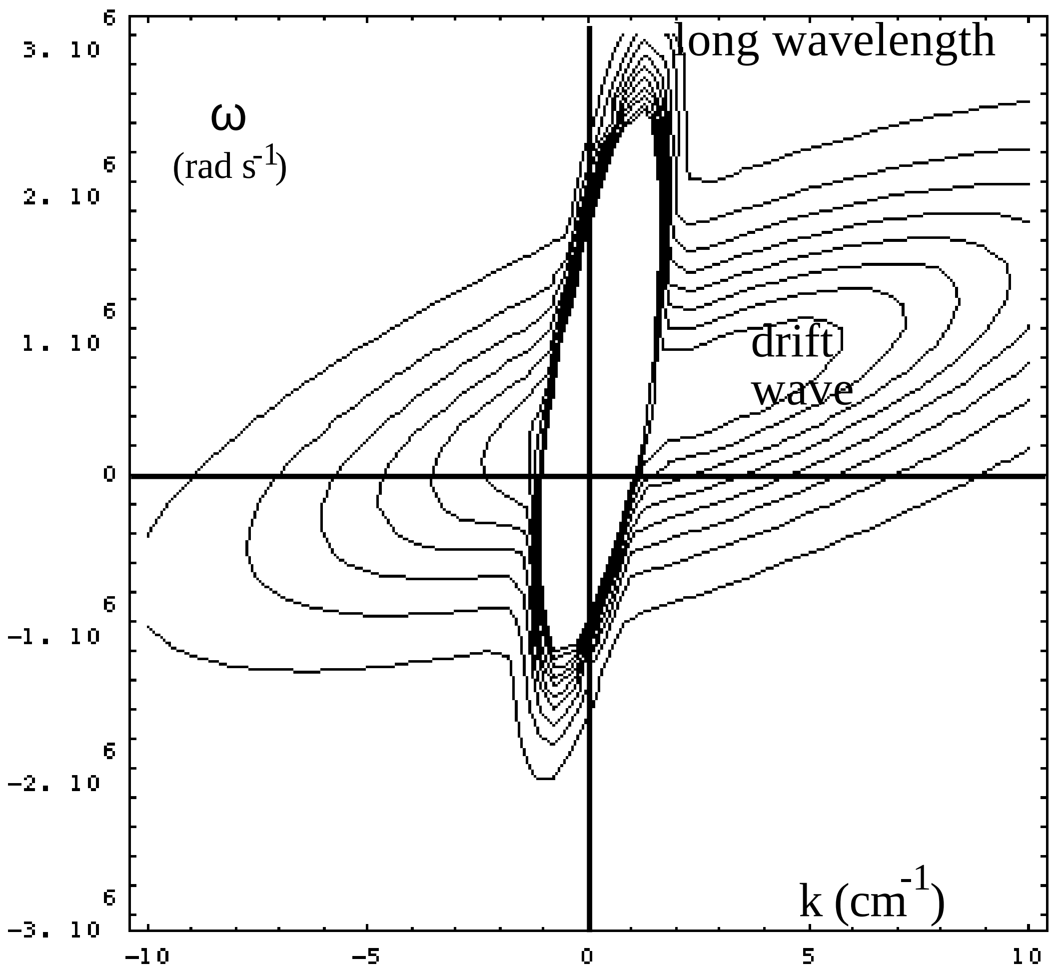
Consider the implications for some particular situations. We consider two "features": 1 is a long wavelength low k feature, and 2 is a more "drift wave" like feature. For our sample volumes we take perfect poloidal alignment (α = 0, δx = 0), and worry only about poloidal effects (i.e. set a = 0).

TURBULENCE

	Feature 1	**Feature 2**
type	long wavelength	drift wave
k_y (cm^{-1})	0.3	3
k_x (cm^{-1})	0.3	3
l_y (cm)	1/0.3	1/3
l_x (cm)	1/0.3	1/3
$S(\omega)$	1	1
$2\sigma_k/k=$	$2\sqrt{2}$	$2\sqrt{2}$

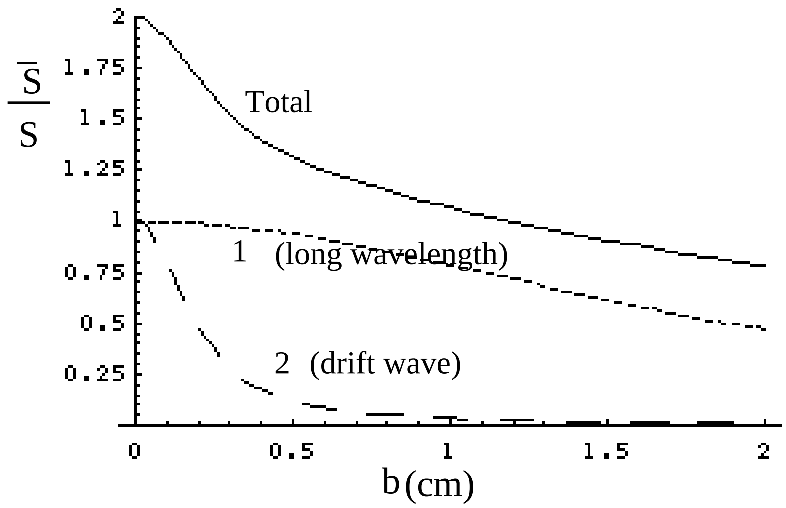
SAMPLE VOLUMES

α	0
a	0
b	variable
δx	0
δy	1.5 cm


long wavelength
ω
(rad s-1)
drift
wave
k (cm-1)
3. 10^6
2. 10^6
1. 10^6
0
-1. 10^6
-2. 10^6
-3. 10^6
-10
-5
0
5
10

Power

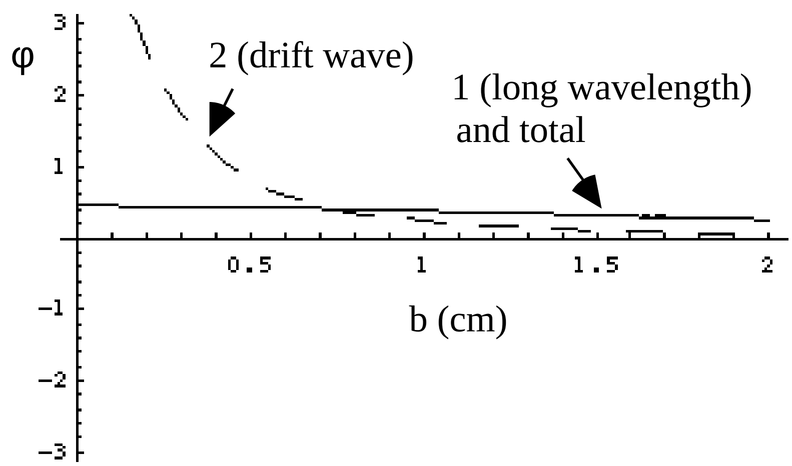
The long wavelength is easily observed with large sample volumes, but the power in the drift wave like feature is only seen as b is decreased (because of the short wavelength)



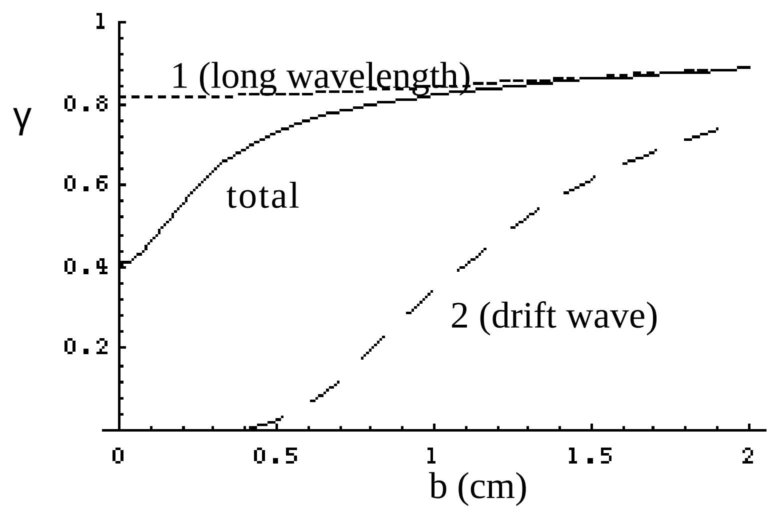
A typical HIBP b = 0.5 cm. implies only 0.6 to 0.7 of the total power is measured.

Phase, wave vector

Only the long wavelength component is picked up. This is because the short wavelength drift wave has such a short correlation length (0.33 cm) that the chosen separation (1.5 cm) is too large for it to be seen. Note the exact phases (b = 0) should be 0.5 (long wavelength) and 4.5 (drift wave)



Coherence, correlation length



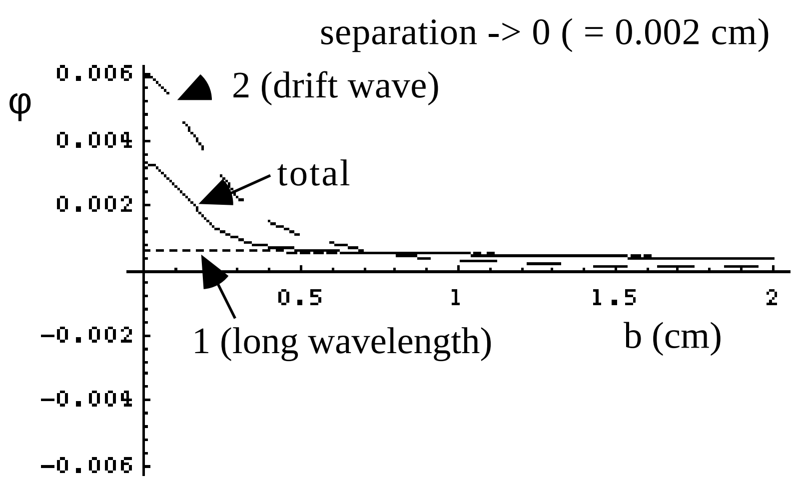
Even for a small sample volume, only an “average” is seen.

What happens with increasing sample volume size is a mess.

Consider poloidal separation 0 (actually δy = 0.002 cm). This will not affect the measured power.

If only a single feature was present, then while the measured phase is affected by the choice of separation, the deduced wave number k is not (but k is affected by the choice of sample volume size b).

In the presence of two features the measured phase is affected by the choice of separation, because of coherence differences. The individual phases would be accurately measured at small b, but the measured phase lies between the two individual phases. That is, the diagnostic would measure an effective k between those actually present.



There are obvious problems with using “large” sample volume size and separations ($b \approx 2\pi/k$, $\delta y \approx l_c$) in that for single features:

large sample volume size gives

- wrong power,
- wrong phase or wave vector,
- wrong coherence or correlation length

large separation is not a problem

In the presence of two features there are other problems.

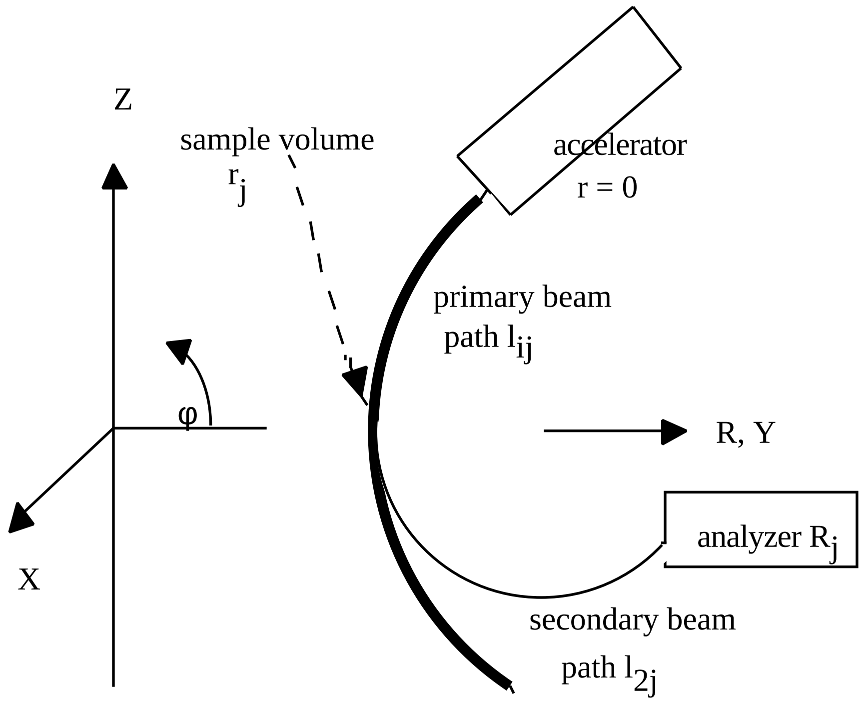
The power is underestimated unless $b \approx 1/k_{max}$ is used. Even if b and δy are chosen small, ($b \approx 1/k_{max}$, $\delta y << l_c$) only an “average” k and correlation length will be measured.

We can use the sensitivities to sample volume size b and separation δy to our advantage: we can filter out the effects of a high wave vector (short wavelength), low correlation length mode by choosing

$$\frac{1}{k_{\mathbf{min}}} >> b >> \frac{1}{k_{\mathbf{max}}} \qquad l_{\mathbf{max}} >> \delta y >> l_{\mathbf{min}}$$

LINE INTEGRALS,COMMON MODE, PATH EFFECTS

Problem: to evaluate a line integral of a fluctuating parameter.



$$I_j = 2 I_{inj}\, \mathbf{exp}\left(-\int_0^{r_j} n_e \sigma_1 dl_1\right) n_e \sigma_{12} \delta l\, \mathbf{exp}\left(-\int_{r_j}^{R_j} n_e \sigma_2 dl_{2j}\right)$$

$$\frac{\tilde{I}_{j\omega}}{I_{0j}} = \frac{\tilde{n}_{e\omega}(r_j)}{n_{e0}(r_j)} - \left(\int_0^{r_j} \tilde{n}_{e\omega} \sigma_1 dl_1 + \int_{r_j}^{R_j} \tilde{n}_{e\omega} \sigma_2 dl_{2j}\right)$$

"BLOBS"

Consider the mean square of some dimensionless parameter Y which varies with density n so that

$$\tilde{Y} = \sigma \int_0^L \tilde{n}\, dx$$

The line integral is along the path 0 to L in the x direction. The square is given by (using dummy variables x_1 and x_2)

$$\tilde{Y}^2 = \sigma^2 \int_0^L \int_0^L \tilde{n}_1(x_1)\tilde{n}_2(x_2)\, dx_1 dx_2$$

where $\tilde{n}_1(x_1)$ means the density fluctuation at point x_1. Assume spatial and temporal averages are equivalent (the "ergodic" assumption), i.e. for any parameter a the average is given by

$$\langle a \rangle = \mathbf{lim}_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} a\, dt = \mathbf{lim}_{X \to \infty} \frac{1}{2X} \int_{-X}^{X} a\, dx$$

The mean square value of Y is

$$\left\langle \tilde{Y}^2 \right\rangle = \left\langle \sigma^2 \int_0^L \int_0^L \tilde{n}_1(x_1)\tilde{n}_2(x_2)\,dx_1 dx_2 \right\rangle$$

$$= \sigma^2 \int_0^L \int_0^L \left\langle \tilde{n}_1(x_1)\tilde{n}_2(x_2) \right\rangle dx_1 dx_2$$

$$= \sigma^2 \int_0^L \int_0^L N_{12}\,dx_1 dx_2$$

$$= \sigma^2 \left\langle \tilde{n}^2 \right\rangle \int_0^L \int_0^L N(x_2 - x_1)\,dx_1 dx_2$$

where the correlation function is

$$N_{12} = \left\langle \tilde{n}_1(x_1,t)\tilde{n}_2(x_2,t) \right\rangle$$

$$= \left\langle \tilde{n}_1(x_1,t)\tilde{n}_2(x_1+\Delta,t) \right\rangle$$

$$= N_{12}(x_2 - x_1)$$

and

$$N(x_2,x_1) = N(x_2 - x_1) = \frac{N_{12}}{\left\langle \tilde{n}^2 \right\rangle}$$

For a spatially homogeneous process the correlation function depends only on the coordinate difference (x_2 - x_1), not on the actual coordinate itself, so that $N(x_2,x_1) = N(x_2 - x_1) = N(X)$, where we have introduced the relative coordinate $X = x_2 - x_1$.

Now choose a functional form for N(X) such as

$$N(X) = e^{-\left(\frac{X}{l_c}\right)^2}$$

where l_c is a correlation length in the coordinate along the path, i.e. in the x direction. If $l_c << L$ then we can replace the integral from -L to L by an integral from -• to • when integrating N(X),

$$\left\langle \tilde{Y}^2 \right\rangle = \sigma^2 \left\langle \tilde{n}^2 \right\rangle \int_0^L \int_{-\infty}^{\infty} N(X)\,dX\,dX$$

$$= 2\sigma^2 \left\langle \tilde{n}^2 \right\rangle \int_0^L \int_0^{\infty} N(X)\,dX\,dX$$

$$= 2\sigma^2 \left\langle \tilde{n}^2 \right\rangle L \int_0^{\infty} N(X)\,dX$$

For the chosen form for N(X) we finally obtain

$$\left\langle \tilde{Y}^2 \right\rangle = \sigma^2 \left\langle \tilde{n}^2 \right\rangle L l_c \sqrt{\pi}$$

Or for the rms value

$$\tilde{Y}_{rms} = \sqrt{\left\langle \tilde{Y}^2 \right\rangle} = \pi^{\frac{1}{4}} \sigma \tilde{n}_{rms} \sqrt{L l_c}$$

We can look on this as a “random walk” process, in which the number of “steps” or “collisions” is L/l_C i.e. the number of correlation distances along the integral. For such a process we know the final displacement is the displacement associated with one “step” (= $\sigma\tilde{n} l_c$) times the square root of the number of steps (= $\sqrt{(L/l_C)}$). You know this because for a random walk the total rms distance Δ moved in a time T is

$$\Delta = \sqrt{2TD}$$

The diffusion coefficient D is itself given by the basic “collision” process time τ and displacement d as

$$D = \frac{d^2}{2\tau}$$

leading to the result I have quoted

$$\Delta = d\sqrt{\frac{T}{\tau}} = d\sqrt{number\ of\ \ collisions}$$

This simple model then gives the mean square “displacement” as

$$\tilde{\Upsilon}_{rms} = \sigma\tilde{n}_{rms}\sqrt{l_c L}$$

which is pretty close (≈ 33% error) to the correct answer.

Applied to our HIBP case, for each Fourier component,

$$\frac{\tilde{I}_{j\omega}}{I_{0j}} = \frac{\tilde{n}_{e\omega}\left(r_j\right)}{n_{e0}\left(r_j\right)} - \left(\int_0^{r_j} \tilde{n}_{e\omega}\sigma_1 dl_1 + \int_{r_j}^{R_j} \tilde{n}_{e\omega}\sigma_2 dl_{2j} \right)$$

i.e. Neglecting cross terms between local and line integral effects, and profiles

$$\left\langle \left(\frac{\tilde{I}_{j\omega}}{I_{0j}} \right)^2 \right\rangle = \left\langle \left(\frac{\tilde{n}_{e\omega}\left(r_j\right)}{n_{e0}\left(r_j\right)} \right)^2 \right\rangle + \left\langle \tilde{Y}^2 \right\rangle$$

$$= \left(\frac{\tilde{n}_{e\omega}}{n_{e0}} \right)^2 + \sqrt{\pi}\sigma^2 \tilde{n}_{e\omega}^2 L l_c$$

$$= \left(\frac{\tilde{n}_{e\omega}}{n_{e0}} \right)^2 \left[1 + \sqrt{\pi}\sigma^2 n_{e0}^2 L l_c \right]$$

i.e. the measured rms quantity is

$$\left(\frac{\tilde{I}_{j\omega}}{I_{0j}} \right)_{rms} = \left(\frac{\tilde{n}_{e\omega}}{n_{e0}} \right)_{rms} \sqrt{1+g}$$

$$g = \sqrt{\pi}\sigma^2 n_{e0}^2 L l_c$$

APPLICATION OF GAUSSIAN STATISTICS TO LINE INTEGRAL EFFECTS

Substitute Gaussian into expression for cross power. Write result in general form. Then

$$\tilde{n}_{e,rms,m} = \tilde{n}_{e,rms}\sqrt{1+g}$$

$$\bar{\gamma}_m = \frac{\left[\bar{\gamma}^2 + g^2 + 2g\bar{\gamma}\,\mathbf{cos}\left(\bar{\varphi}\right)\right]^{1/2}}{1+g}$$

$$\bar{\varphi}_m = \mathbf{tan}^{-1}\left(\frac{\bar{\gamma}\,\mathbf{sin}\left(\bar{\varphi}\right)}{\bar{\gamma}\,\mathbf{cos}\left(\bar{\varphi}\right) + g}\right)$$

For a simple case (equal input and output beams, flat profiles)

$$g = -2\sqrt{\pi}\left(n_e \sigma L\right)\frac{l_c}{L}\, e^{-\left(\frac{\bar{k}_y l_c}{2}\right)^2}$$

$$+2\sqrt{\pi}\left(n_e \sigma L\right)^2 \frac{l_c}{L}\, e^{-\left(\frac{\bar{k}_y l_c}{2}\right)^2}$$

Typically g = 0.2.